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Some results on extension of lattice-valued QL-implications

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Abstract

Background: A very important issue in lattice theory is how to extend a given operator preserving its algebraic properties. For lattice-valued fuzzy operators framework, in 2008 Saminger-Platz presented a way to extend t-norms which was generalized by Palmeira et al. (2011) for t-norms, t-conorms, fuzzy negations and implications, considering the scenery provided by the (r,s)-sublattice.

Methods: In this paper we investigated how to extend QL-implications and which properties of it are preserved by the extension method via retractions (EMR).

Results: As results, we proved that properties (LB), (RB), (CC1), (CC2), (CC3), (CC4), (L-NP), (EP) and (IP) are preserved by EMR.

Conclusions: However, the extension method via retractions fails in preserving the important properties (NP), (OP), (IBL), (CP), (P) and (LEM).

Keywords: QL-implication, Extension, (r,s)-sublattice, Lattice

Background

Let L and K be nonempty sets and suppose that M is a subset of L. Given a function $f : M \longrightarrow K$, if we want to extend the domain of f to cover the whole L, what is the best choice to define $f(x)$ for the elements $x \in L\backslash M$? The answer is: it depends! This is very simple if we want only to construct a new function that has L as its domain. In this case, it is enough, for example, to define $f(x) = a$ for a suitable and fixed a belonging to K (i.e., define f as a constant function for the elements belonging to $L\backslash M$). However, this task becomes more complex if we want to preserve some characteristics and properties of f.

In fuzzy logic, the problem of extending functions can be considered for lattice-valued fuzzy connectives (t-norms, t-conorms, negations, and others) since these connectives are functions, in particular. The pioneer work in this framework was put forward by Saminger-Platz et al. in [1] which provides a method to extend a t-norm T from a complete sublattice M to a bounded lattice L. Later, we have developed in [2] an extension method to extend t-norms, t-conorms, and fuzzy negations that generalizes the method proposed in [1] considering a modified notion of sublattice. Also, we have applied this method for fuzzy implications in [3].

The class of QL-implications is the generalization for fuzzy logic of the implications of quantum logic which raised from the Garrett Birkhoff and John von Neumann conclusion that "propositional calculus of quantum mechanics has the same structure as an abstract projective geometry." It opened the way for the development algebraic logic that have much weaker properties than Boolean algebras. Another interesting fact is that projective geometry is a non-distributive modular lattice.

In this work, we apply the extension method developed in [2] for QL-implications. To do so, we recall some elementary concepts related to lattice theory in Section "Background and literature review." The extension method via retractions is presented in Section "Research design and methodology," for t-norms, t-conorms, fuzzy negations, and implications. Section "Methods" is devoted to present the main results of this paper, namely the results concerning to the extension of QL-implications.

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Background and literature review

Lattice-valued fuzzy logic and related theories have been studied by many researchers since lattice provides a very good scenery for the real world issues. For example, in mathematical morphology, lattice appeals to integral geometry, stereolgy, and random set models; it is mainly its algebraic facet which has become popular. There are also many other applications for lattice in image processing. So it is essential to have a very consistent mathematical theory in order to provide a safe framework to deal with those issues (see [4, 5]).

In this paper, we rise up a discussion on the lattice-valued QL-implications and its algebraic extension as a function. To do so, in which follows, we provide a review on some important definitions and results.

Bounded lattices: definition and related concepts

We consider here the algebraic notion of lattices the reasons for this choice will be clear from the context. But a discussion about the other approach to lattices (i.e., as posets) can be found in [6–8].

Definition 1 *Let L be a nonempty set. If $\wedge_L$ and $\vee_L$ are two binary operations on L, then $\langle L, \wedge_L, \vee_L\rangle$ is a lattice provided that for each $x, y, z \in L$, the following properties hold:*

1. $x \wedge_L y = y \wedge_L x$ *and* $x \vee_L y = y \vee_L x$ *(symmetry);*
2. $(x \wedge_L y) \wedge_L z = x \wedge_L (y \wedge_L z)$ *and* $(x \wedge_L y) \vee_L z = x \vee_L (y \wedge_L z)$ *(associativity);*
3. $x \wedge_L (x \vee_L y) = x$ *and* $x \vee_L (x \wedge_L y) = x$ *(distributivity).*

If in $\langle L, \wedge_L, \vee_L\rangle$ there are elements 0_L and 1_L such that, for all $x \in L$, $x \wedge_L 1_L = x$ and $x \vee_L 0_L = x$, then $\langle L, \wedge_L, \vee_L, 0_L, 1_L\rangle$ is called a *bounded lattice*. Moreover, it is known that, given a lattice L, the relation $x \leq_L y$ if and only if $x \wedge_L y = x$ defines a partial order on L. Recall also that a lattice L is called a complete lattice if every subset of it has a supremum and an infimum element[1].

Example 1 *The set* $[0, 1]$ *endowed with the operations defined by* $x \wedge y = \min\{x, y\}$ *and* $x \vee y = \max\{x, y\}$ *for all* $x, y \in [0, 1]$ *is a (complete) bounded lattice in the sense of Definition* 1 *which has 0 as the bottom and 1 as the top element.*

Remark 1 *In order to simplify the notation, throughout this paper when we say that L is a bounded lattice, it means that L has a structure as in Definition* 1*.*

Definition 2 *Let* $(L, \wedge_L, \vee_L, 0_L, 1_L)$ *and* $(M, \wedge_M, \vee_M, 0_M, 1_M)$ *be bounded lattices. A mapping* $f : L \longrightarrow M$ *is said to be a lattice homomorphism if, for all $x, y \in L$, we have*

1. $f(x \wedge_L y) = f(x) \wedge_M f(y)$
2. $f(x \vee_L y) = f(x) \vee_M f(y)$
3. $f(0_L) = 0_M$ *and* $f(1_L) = 1_M$*.*

Remark 2 *Recall that, an injective (a surjective) lattice homomorphism is called a monomorphism (epimorphism) and a bijective lattice homomorphism is called an isomorphism. An automorphism is an isomorphism from a lattice onto itself.*

Proposition 1 [2] *Every lattice homomorphism preserves the order.*

Proposition 2 [9] *Let L be a bounded lattice. Then, a function $\rho : L \longrightarrow L$ is an automorphism if and only if* (1) *ρ is bijective and* (2) *$x \leq_L y$ if and only if $\rho(x) \leq_L \rho(y)$.*

From now on, lattice homomorphisms will be called just homomorphisms for simplicity.

Definition 3 *Given a function $f : L^n \rightarrow L$, the action of an L-automorphism ρ over f results in the function $f^\rho : L^n \rightarrow L$ defined as*

$$f^\rho(x_1, \ldots, x_n) = \rho^{-1}(f(\rho(x_1), \ldots, \rho(x_n))) \quad (1)$$

In this case, f^ρ is said to be a conjugate of f (see [10]*).*

Let $f : L^n \rightarrow L$ be a conjugate of $g : L^n \rightarrow L$. If $f(x_1, \ldots, x_n) \leq_L g(x_1, \ldots, x_n)$ for each $x_1, \ldots, x_n \in L$ then we denote it by $f \leq g$.

Retracts and sublattices

In general, given a bounded lattice L and a nonempty subset $M \subseteq L$, it is said that M is a sublattice of L if, for all $x, y \in M$, the following conditions hold: $x \wedge_L y \in M$ and $x \vee_L y \in M$. In other words, M equipped with the restriction of the operations $\wedge_L$ and $\vee_L$ inherits the lattice structure of L.

We would like to work in a generalized notion of sublattice in which the condition $M \subseteq L$ is somewhat weakened.

Definition 4 [11] *A homomorphism r of a lattice L onto a lattice M is said to be a retraction if there exists a homomorphism s of M into L which satisfies $r \circ s = id_M$. A lattice M is called a retract of a lattice L if there is a retraction r, of L onto M, and s is then called a pseudo-inverse of r.*

Definition 5 *Let L and M be arbitrary bounded lattices. We say that M is a (r, s)-sublattice of L if M is a retract of L (i.e., M is a sublattice of L up to isomorphisms). In other* *words, M is a (r,s)-sublattice of L if there is a retraction r of L onto M with pseudo-inverse $s : M \to L$.*

The purpose of defining (r,s)-sublattices as done in Definition 5 is to provide a relaxed notion of this concept. It is done an identification of M with a subset $K = s(M)$ of L in order to carry on some properties of M to K, including its lattice structure via retraction r. In this case, K works as an algebraic copy of M embedded into L since r is a homomorphism.

Remark 3 *Throughout this paper, the concept of (r,s)-sublattice as in Definition 5 is used. Whenever the usual definition of sublattice is used and this is not clear from the context, this sublattice will be called ordinary sublattice.*

The main advantage behind the idea of using this relaxed version of sublattice is that it allows us to verify the validity for L of a property which is invariant under homomorphisms from a lattice M without requiring M be a subset of L.

Definition 6 *Every retraction $r : L \longrightarrow M$ (with pseudo-inverse s) which satisfies $s \circ r \leq id_L{}^2$ ($id_L \leq s \circ r$) is called a lower (an upper) retraction. In this case, M is called a lower (an upper) retract of L.*

Notice that both in Definitions 5 and 6, the pseudo-inverse s of a retraction r cannot be unique. This is an advantage of our notion of sublattice since if there exist more than one pseudo-inverse for the same retraction, it is possible to identify M with a subset of L in different ways what give us the possibility to choose the best one for our proposes. But we must be clear that when we say that M is a (lower, upper or neither) (r,s)-sublattice of L, we are considering the existence of at least one pseudo-inverse s and fixing it. No matter which pseudo-inverse is taken, every result presented here remains working.

Example 2 *Let M and L be bounded lattices as shown in Fig. 1. A mapping $r : L \longrightarrow M$ given by $r(x) = \sup\{z \in M \mid s(z) \leq_L x\}$ is a lower retraction whose pseudo-inverse is the mapping $s : M \longrightarrow L$ defined by $s(1_M) = 1_L$, $s(a) = v$, $s(b) = x$, $s(c) = y$, $s(d) = z$ and $s(0_M) = 0_L$. Therefore, it follows that M is a (r,s)-sublattice of L in the sense of Definition 5.*

Remark 4 *Note that given a lower retraction, it is sometimes possible to define an upper retraction with the same pseudo-inverse. For instance, let L and M be lattices as shown in Fig. 1. If r is a lower retraction with pseudo-inverse s as defined in the Example 2, then the function r' given by $r'(x) = \inf\{z \in M \mid s(z) \geqslant_L x\}$ is an upper retraction since $id_L \leqslant s \circ r'$. It is easy to check that s is also a pseudo-inverse of r'.*

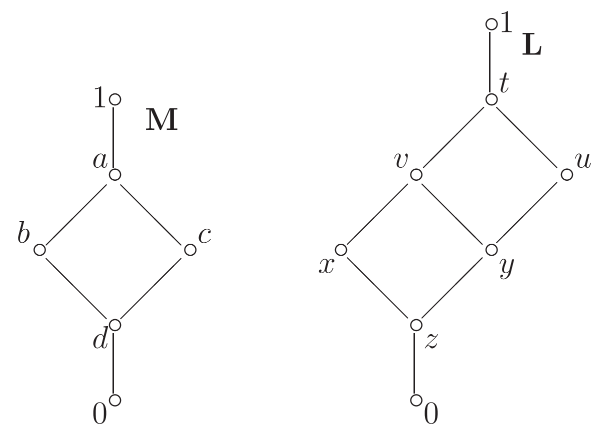

Fig. 1 Hasse diagrams of lattices *M* and *L*

It is worth noting that if M is a (r,s)-sublattice of L then there is a retraction r from L onto M, but it is not required to r to be a lower or an upper retraction. Nevertheless, as shown in the Remark above, there may be more than one retraction from L onto M with the same pseudo-inverse. This is a very useful particularity of Definition 5 and we would like to highlight it in a definition.

Definition 7 *Let M be a (r_1,s)-sublattice of L. We say that*

1. *M is a lower (r_1,s)-sublattice of L if r_1 is a lower retraction. Notation: $M < L$ with respect to (r_1,s)*
2. *M is an upper (r_1,s)-sublattice of L whenever r_1 is an upper retraction. Notation: $M > L$ with respect to (r_1,s)*
3. *If r_1 is a lower retraction and there is an upper retraction $r_2 : L \longrightarrow M$ such that its pseudo-inverse is also s, then M is called a full (r_1,r_2,s)-sublattice of L. Notation: $M \trianglelefteq L$ with respect to (r_1,r_2,s).*

Remark 5 *Let L be a complete bounded lattice. We define the case when M is a complete and lower (respectively upper) (r,s)-sublattice of L by $M \lessdot L$ (by $M \gtrdot L$).*

An immediate consequence of the definition of lower (upper) retraction is that if $M \trianglelefteq L$ then it follows that $s \circ r_1 \leqslant id_L \leqslant s \circ r_2$.

Fuzzy connectives

In which follows, we define some well-known interpretation of the classical connectives in lattice-valued fuzzy logic [12–14].

Definition 8 *Let L be a bounded lattice. A binary operation $T(S) : L \times L \longrightarrow L$ is a t-norm (t-conorm) if, for all $x, y, z \in L$, it satisfies:*

1. $T(x,y) = T(y,x)$ $(S(x,y) = S(y,x))$ *(commutativity)*
2. $T(x, T(y,z)) = T(T(x,y),z)$ $(S(x, S(y,z)) = S(S(x,y),z))$ *(associativity)*
3. *If* $x \leqslant_L y$ *then* $T(x,z) \leqslant_L T(y,z)$ $(S(x,z) \leqslant_L S(y,z))$, $\forall z \in L$ *(monotonicity)*
4. $T(x, 1_L) = x$ $(S(x, 0_L) = x)$ *(boundary condition).*

Dually, it is possible to define the concept of t-conorms.

Definition 9 *Let L be a bounded lattice. A binary operation* $S : L \times L \longrightarrow L$ *is said be a t-conorm if, for all* $x, y, z \in L$*, we have:*

1. $S(x,y) = S(y,x)$ *(commutativity)*
2. $S(x, S(y,z)) = S(S(x,y),z)$ *(associativity)*
3. *If* $x \leqslant_L y$ *then* $S(x,z) \leqslant_L S(y,z)$, $\forall z \in L$ *(monotonicity)*
4. $S(x, 0_L) = x$ *(boundary condition).*

Definition 10 *A function* $N : L \longrightarrow L$ *is called a fuzzy negation if it satisfies:*

(N1) $N(0_L) = 1_L$ *and* $N(1_L) = 0_L$
(N2) If $x \leqslant_L y$ *then* $N(y) \leqslant_L N(x)$*, for all* $x, y \in L$*.*

Moreover, the negation N is strong if it also satisfies the involution property, namely

(N3) $N(N(x)) = x$, for all $x \in L$

In case N satisfies

(N4) $N(x) \in \{0_L, 1_L\}$ if and only if $x = 0_L$ or $x = 1_L$,

it is called frontier. In addition, every element $x \in L$ such that $N(x) = x$ is said to be an equilibrium point of N.

From the point of view of lattice theory, a strong negation corresponds to what is known as involution (see [6]).

Definition 11 *Let T be a t-norm on the complete lattice L. The function* $N_T : L \to L$ *given by*

$$N_T(x) = \sup\{y \in L \mid T(x,y) = 0_L\} \tag{2}$$

is a fuzzy negation, called natural negation of T or the negation induced by T.

Similarly, we can define a natural negation of a t-conorm S as follows.

Definition 12 *Let S be a t-conorm on the complete lattice L. The function* $N_S : L \to L$ *given by*

$$N_S(x) = \inf\{y \in L \mid S(x,y) = 1_L\} \tag{3}$$

is a fuzzy negation, called natural negation of S or the negation induced by S.

Proposition 3 *Let T be a t-norm and S be a t-conorm on complete lattice L. Thus*

1. *if* $T(x,y) = 0_L$ *for some* $x, y \in L$ *then* $y \leqslant N_T(x)$
2. *if* $S(x,y) = 1_L$ *for some* $x, y \in L$ *then* $y \geqslant N_S(x)$
3. *if* $z < N_T(x)$ *for some* $x, y \in L$ *then* $T(x,z) = 0_L$
4. *if* $z > N_S(x)$ *for some* $x, y \in L$ *then* $S(x,z) = 1_L$

Proof Similar to Remark 2.3.2(iii) of [15]. □

Finally, we present the notion of fuzzy implication. There are some different interpretations of this fuzzy operator in the literature (see [15–20]) since there is no consensus on the way to define it just that fuzzy implication have to behavior at least as in the crisp case. Here, we consider the notion presented in [15] because we believe such a definition has the properties necessary for a fuzzy implication.

Definition 13 *A function* $I : L \times L \longrightarrow L$ *is a fuzzy implication on L if for each* $x, y, z \in L$ *the following properties hold:*

1. *(FPA) if* $x \leqslant_L y$ *then* $I(y,z) \leqslant_L I(x,z)$ *(first place antitonicity)*
2. *(SPI) if* $y \leqslant_L z$ *then* $I(x,y) \leqslant_L I(x,z)$ *(second place isotonicity)*
3. *(CC1)* $I(0_L, 0_L) = 1_L$ *(corner condition 1)*
4. *(CC2)* $I(1_L, 1_L) = 1_L$ *(corner condition 2)*
5. *(CC3)* $I(1_L, 0_L) = 0_L$ *(corner condition 3)*

Consider also the following properties of an implication I on L:

(LB) $I(0_L, y) = 1_L$, for all $y \in L$
(RB) $I(x, 1_L) = 1_L$, for all $x \in L$
(CC4) $I(0_L, 1_L) = 1_L$
(NP) $I(1_L, y) = y$ for each $y \in L$ (left neutrality principle)
(L-NP) $I(1_L, y) \leqslant_L y$ for each $y \in L$
(EP) $I(x, I(y,z)) = I(y, I(x,z))$ for all $x, y, z \in L$ (exchange principle)
(IP) $I(x,x) = 1_L$ for each $x \in L$ (identity principle)
(OP) $I(x,y) = 1_L$ if and only if $x \leqslant y$ (ordering property)
(IBL) $I(x, I(x,y)) = I(x,y)$ for all $x, y, z \in L$ (iterative Boolean law)
(CP) $I(x,y) = I(N(y), N(x))$ for each $x, y \in L$ with N a fuzzy negation on L (law of contraposition)
(L-CP) $I(N(x), y) = I(N(y), x)$ (law of left contraposition)

(R-CP) $I(x, N(y)) = I(y, N(x))$ (law of right contraposition)
(P) $I(x, y) = 0_L$ if and only if $x = 1_L$ and $y = 0_L$ (Positivety)
(LEM) $S(N(x), x) = 1_L$ for each $x \in L$ (law of excluded middle)

Note that, a special class of fuzzy implication can be naturally obtained by generalizing the implication operator from the quantum logic, namely $p \to q \Leftrightarrow \neg p \vee (p \wedge q)$. For bounded lattices, this implication is given as follows.

Definition 14 *A function $I : L \times L \to L$ is called a QL-implication if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that*

$$I(x, y) = S(N(x), T(x, y)) \tag{4}$$

for all $x, y \in L$, is a fuzzy implication. If I is a QL-implication generated by the triple $\langle T, S, N\rangle$, then we will often denote it by $I_{T,S,N}$.

Remark 6 [15] *Notice that not all function I satisfying Eq. (4) is a fuzzy implication. For instance, if $L = [0, 1]$, T is the drastic t-norm, i.e.,*

$$T(x, y) = \begin{cases} x, \ y = 1; \\ y, \ x = 1; \\ 0, \ otherwise. \end{cases}$$

then the function

$$I_{T,S,N}(x, y) = \begin{cases} 1, \ y = 1; \\ y, \ x = 1; \\ N(x), \ \text{otherwise}. \end{cases}$$

is not always a fuzzy implication, even if S and N satisfy (LEM).

Definition 15 *Let T, S, and N be a t-norm, a t-conorm, and fuzzy negation on L, respectively. Then, the function $N_{I_{T,S,N}} : L \to L$ given by*

$$N_{I_{T,S,N}}(x) = I_{T,S,N}(x, 0_L) \tag{5}$$

for all $x \in L$ is a fuzzy negation. Usually $N_{I_{T,S,N}}$ is called the natural negation generated from $I_{T,S,N}$.

Proposition 4 [15] *Let T be a t-norm, S a t-conorm, and N a fuzzy negation defined on L. Then*

1. *$I_{T,S,N}$ satisfies (SPI), (CC1), (CC2), (CC3), (CC4), (LB), and (NP)*
2. *$N_{I_{T,S,N}} = N$*

Proposition 5 [15] *If $I_{T,S,N}$ is a QL-implication, then the conjugate of $I_{T,S,N}$ is also a QL-implication generated from the conjugate of T, S and N, i.e.,*

$$(I_{T,S,N})^\rho = I_{T^\rho, S^\rho, N^\rho}$$

Research design and methodology

As explained at the beginning, the main goal of this paper is to provide a discussion about the extension of lattice-valued QL-implications applying the method proposed in [2] in order to verify which properties are preserved by the extension operator. Because it is a theoretical research the methodology is basically to state and prove results.

Methods

We start this section presenting the extension method developed in [2, 21] for t-norms, t-conorms, and fuzzy negations. Also in this framework, we apply this method for extending QL-implications considering our previous study about extension of fuzzy implications described in [3].

Extension method via retractions (EMR)

We start this section presenting the extension method developed in [2] for t-norms, t-conorms and fuzzy negations. Also in this framework, we apply this method for extending QL-implications considering our previous study about extension of fuzzy implications described in [3].

Consider an ordinary sublattice M of a bounded lattice L (i.e., $M \subseteq L$) and T a t-norm on M. Since a t-norm is particularly a function, it is natural to think if it is possible to extend T from M to L in order to obtain a new t-norm T^E on L.

One of the first published works on this subject was put forward by Saminger-Platz et al. in [1]. There, it was proposed a method to extend a given t-norm T defined on a complete ordinary sublattice M of lattice L.

Seeking to generalize this extension method considering the relaxed notion of sublattice as in Definition 5, Palmeira and Bedregal presented in [2] other way to extend t-norms, t-conorms, and fuzzy negations as we can see in the following propositions.

Proposition 6 [2] *Let $M < L$ with respect to (r, s). If T is a t-norm on M then $T^E : L \times L \longrightarrow L$ defined by*

$$T^E(x, y) = \begin{cases} x \wedge_L y, & if \ 1_L \in \{x, y\} \\ s(T(r(x), r(y))), & otherwise. \end{cases} \tag{6}$$

is a t-norm which extends T from M to L.

It is also possible to apply the method of extending t-norms for t-conorms and fuzzy negations under similar conditions as one can see in Propositions 7 and 8 below.

Proposition 7 [2] *Let $M > L$ with respect to (r, s). If S is a t-conorm on M, then $S^E : L \times L \longrightarrow L$ defined by*

$$S^E(x,y) = \begin{cases} x \vee_L y, & \text{if } 0_L \in \{x,y\} \\ s(S(r(x), r(y))), & \text{otherwise.} \end{cases} \quad (7)$$

is a t-conorm which extends S from M to L.

Corollary 1 [2] *Let $M > L$ with respect to (r, s), ρ be an automorphism on M and T be a t-norm on M. Moreover, suppose $\psi : L \longrightarrow L$ is an automorphism on L such that $r \circ \psi = \rho \circ r$. Then, $(S^\rho)^E \geqslant (S^E)^\psi$.*

Proposition 8 [2] *Let M be a (r, s)-sublattice of L and $N : M \longrightarrow M$ be a fuzzy negation. Then*

$$N^E(x) = s(N(r(x))) \quad (8)$$

for each $x \in L$ is a fuzzy negation that extends N from M to L.

It is worth noting that in Proposition 8, it is required only that r needs to be a retraction (it is not necessary to be neither a lower nor an upper retraction), and hence if r is a lower retraction or an upper retraction, the result remains valid. This fact allows us to extend fuzzy negations in a more flexible way than t-norms and t-conorms.

Proposition 9 *Let T be a t-norm and S be a t-conorm on lattice M. Thus*

1. *if $M \lessdot L$ with respect (r_1, s) and $T^E(x,y) = 0_L$ for some $x, y \in L$ then $s(r_1(y)) \leqslant N^E_T(x)$*
2. *if $M \gtrdot L$ with respect (r_2, s) and $S^E(x,y) = 1_L$ for some $x, y \in L$ then $s(r_2(y)) \geqslant N^E_S(x)$*
3. *if $M \lessdot L$ and $z < N^E_T(x)$ for some $x, z \in L$ then $T^E(x,z) = 0_L$*
4. *if $M \gtrdot L$ and $z > N^E_S(x)$ for some $x, z \in L$ then $S^E(x,z) = 1_L$*

Proof 1. Suppose $T^E(x,y) = 0_L$ e $N^E_T(x) = s(N_T(r_1(x)))$ for each $x \in L$. Hence, if $x = 1_L$ or $y = 1_L$, we have that $T^E(x,y) = x \wedge y = 0_L$. Without loss of generality, put $x = 1_L$ and then $y = 0_L$. Therefore, $0_L = s(r_1(y)) < N^E_T(x) = 1_L$. On the other hand, if $x \neq 1_L$ and $y \neq 1_L$ then

$$s(T(r_1(x), r_1(y))) = 0_L \Rightarrow T(r_1(x), r_1(y)) = r(0_L) = 0_M \Rightarrow$$
$$\Rightarrow r_1(y) \leqslant N_T(r_1(x)) \Rightarrow s(r_1(y)) \leqslant N^E_T(x).$$

2. For this item, we take $N^E_S(x) = s(N_S(r_2(x)))$ for each $x \in L$. If $S^E(x,y) = 1_L$ for $x \neq 0_L$ and $y \neq 0_L$ then $s(S(r_2(x), r_2(y))) = 1_L$. Since r_1 is order-preserving, it follows that $S(r_2(x), r_2(y)) = r_1(1_L) = 1_M$, and by Proposition 3 item 2, $r_2(y) \geqslant N_S(r_2(x))$ which implies $s(r_2(y)) \geqslant s(N_S(r_2(x))) = N^E_S(x)$. If $x = 0_L$ or $y = 0_L$, then supposing $x = 0_L$, we have that $1_L = S^E(x,y) = x \vee y$ and hence $y = 1_L$. Therefore, $s(r_2(1_L)) = 1_L = N^E_S(0_L)$.
3. If $z < N^E_T(x) = s(N_T(r_1(x)))$ then $r_1(z) \leqslant N_T(r_1(x))$. By Proposition 3 item 3, we have $T(r_1(x), r_1(z)) = 0_M$ and hence $s(T(r_1(x), r_1(z))) = s(0_M) = 0_L$. Thus $T^E(x,z) = 0_L$.
4. Consider N^E_S as defined at item 2. If $z > N^E_S(x) = s(N_S(r_2(x)))$ then $r_2(z) \geqslant N_S(r_2(x))$. Again by Proposition 3, it follows that

$$S(r_2(x), r_2(z)) = 1_M \Rightarrow s(S(r_2(x), r_2(z)) = s(1_M) = 1_L \Rightarrow S^E(x,z) = 1_L$$

□

The following theorem presents a way to extend fuzzy implications by applying the method of extending fuzzy operators as introduced in [2].

Theorem 1 [3] *Let M be a (r, s)-sublattice of L. If I is an implication on M then the function $I^E : L \times L \longrightarrow L$ given by*

$$I^E(x,y) = s(I(r(x), r(y))) \quad (9)$$

for all $x, y \in L$, is an implication on L. In this case, I^E is called the extension of I from M to L.

Proposition 10 [3] *Under the same conditions as in Theorem 1, if I is an implication on M satisfying some of properties (LB), (RB), (CC4), (EP), (IP), (IBL), and (CP), then I^E is an implication on L which satisfies the same properties.*

Proposition 11 [3] *Let M be a (r, s)-sublattice of L, ρ be an automorphism on M and I be an implication on M. Moreover, suppose $\psi : L \to L$ is an automorphism on L such that $r \circ \psi = \rho \circ r$ and $\psi^{-1} \circ s = s \circ \rho^{-1}$. Then, $(I^\rho)^E = (I^E)^\psi$.*

Results and discussion

The main results are presented in what follows as well as a critical analysis of them. We start presenting the extension of the QL-implications in the first subsection and then a discussion is done for its extension and properties (EP) and (IP). Finally, a table summarize which properties of QL-implications are or not preserved by the extension method.

Extension of QL-implications

As shown in the previous subsection, the extension method via retractions can be used for extending t-norms, t-conorms, fuzzy negations, and implications. Now, we want to apply this method to extend QL-implications and test which properties related to this operator can be preserved by this extension method.

Theorem 2 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). If $I_{T,S,N}$ is a QL-implication on M, then the function given by $I^E_{T,S,N}(x, y) = s(I_{T,S,N}(r_1(x), r_1(y))$ for each $x, y \in L$ is a QL-implication on L.*

Proof Straightforward from Theorem 1. □

Proposition 12 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). If T, S, and N are a t-norm, a t-conorm, and a fuzzy negation respectively defined on M, such that $I_{T,S,N}$ is a QL-implication on M, then the function given by $I_{T^E,S^E,N^E}(x, y) = S^E(N^E(x), T^E(x, y))$ for each $x, y \in L$ is a QL-implication.*

Proof Since r_1 and r_2 are a lower and an upper retractions, respectively, we can extend T with respect to (r_1, s) as in (6) and S with respect to (r_2, s) as in (7). Moreover, for each $x \in L$, $N^E(x) = s(N(r_1(x)))$ is an extension of N from M to L (see Proposition 8). In this case, we have that

$$
\begin{aligned}
I_{T^E,S^E,N^E}(x, y) &= S^E(N^E(x), T^E(x, y)) \\
&= s(S(r_2(N^E(x)), r_2(T^E(x, y)))) \\
&= s(S(r_2(s(N(r_1(x)))), \\
&\quad r_2(s(T(r_1(x), r_1(y)))))) \\
&= s(S(N(r_1(x)), T(r_1(x), r_1(y)))) \\
&= s(I_{T,S,N}(r_1(x), r_1(y)))
\end{aligned}
$$

for all $x, y \in L$.

Considering this fact, we shall prove that I_{T^E,S^E,N^E} satisfies (*FPA*), (*SPI*), (*CC*1), (*CC*2), and (*CC*3).

(FPA)

Let $x, y \in L$ such that $x \leqslant_L y$. Since $I_{T,S,N}$ is a QL-implication (in this case, it satisfies (FPA)) and $r_1(x) \leqslant_M r_1(y)$ then, for all $z \in L$, it follows that

$$
\begin{aligned}
I_{T^E,S^E,N^E}(y, z) &= s(I_{T,S,N}(r_1(y), r_1(z))) \\
&\leqslant_L s(I_{T,S,N}(r_1(x), r_1(z))) \\
&= I_{T^E,S^E,N^E}(x, z)
\end{aligned}
$$

Analogously, it can be proved that I_{T^E,S^E,N^E} satisfies (SPI) since $I_{T,S,N}$ satisfies (SPI).

Moreover,

(CC1)

$$
\begin{aligned}
I_{T^E,S^E,N^E}(0_L, 0_L) &= s(S(N(r_1(0_L)), T(r_1(0_L), r_1(0_L)))) \\
&= s(S(1_M, 0_M)) \\
&= s(1_M) = 1_L
\end{aligned}
$$

(CC2)

$$
\begin{aligned}
I_{T^E,S^E,N^E}(1_L, 1_L) &= s(S(N(r_1(1_L)), T(r_1(1_L), r_1(1_L)))) \\
&= s(S(N(1_M), T(1_M, 1_M))) \\
&= s(1_M) = 1_L
\end{aligned}
$$

(CC3)

$$
\begin{aligned}
I_{T^E,S^E,N^E}(1_L, 0_L) &= s(S(N(r_1(1_L)), T(r_1(1_L), r_1(0_L)))) \\
&= s(S(0_M, 0_M)) \\
&= s(0_M) = 0_L
\end{aligned}
$$

□

Corollary 2 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). If T, S, and N are a t-norm, a t-conorm, and a fuzzy negation, respectively, all of them defined on M, then $I_{T^E,S^E,N^E} = I^E_{T,S,N}$.*

Proof For all $x, y \in L$, it follows that

$$
\begin{aligned}
I_{T^E,S^E,N^E}(x, y) &= S^E(N^E(x) T^E(x, y)) \\
&= s(S(r_2(s(N(r_1(x)))), r_2(s(T(r_1(x), r_1(y)))))) \\
&= s(S(N(r_1(x)), T(r_1(x), r_1(y)))) \\
&= s(I_{T,S,N}(r_1(x), r_1(y)) \\
&= I^E_{T,S,N}(x, y)
\end{aligned}
$$

□

Proposition 13 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). If T is a t-norm, S a t-conorm, and N a fuzzy negation defined on M, respectively, then*

1. *I_{T^E,S^E,N^E} satisfies (SPI), (CC1), (CC2), (CC3), (CC4), (LB), (RB), and (L-NP)*
2. *$N_{I_{T^E,S^E,N^E}} = N^E$*

Proof 1. From Proposition 12, we can conclude that I_{T^E,S^E,N^E} satisfies (SPI), (CC1), (CC2), and (CC3). Moreover, for all $y \in L$ we have that

$$
\begin{aligned}
I_{T^E,S^E,N^E}(0_L, y) &= s(S(r_2(s(N(r_1(0_L)))), r_2(s(T(r_1(0_L), r_1(y)))))) \\
&= s(S(N(r_1(0_L)), T(r_1(0_L), r_1(y)))) \\
&= s(S(1_M, 0_M)) \\
&= s(1_M) = 1_L
\end{aligned}
$$

and

$$
\begin{aligned}
I_{T^E,S^E,N^E}(x, 1_L) &= s(S(r_2(s(N(r_1(x)))), r_2(s(T(r_1(x), r_1(1_L)))))) \\
&= s(S(N(r_1(x)), T(r_1(x), r_1(1_L)))) \\
&= s(S(N(r_1(x)), T(r_1(x), 1_M))) \\
&= s(S(N(r_1(x)), r_1(x))) \\
&= s(1_M) = 1_L
\end{aligned}
$$

what means that I_{T^E,S^E,N^E} satisfies (LB) and (RB). For showing that it satisfies (L-NP), it is enough to see that for each $y \in L$ we have

$$
\begin{aligned}
I_{T^E,S^E,N^E}(1_L, y) &= s(S(N(r_1(1_L)), T(r_1(1_L), r_1(y)))) \\
&= s(S(0_M, r_1(y))) \\
&= s(r_1(y)) \leqslant_L y
\end{aligned}
$$

It is easy to see that I_{T^E,S^E,N^E} satisfies (CC4) since it satisfies (LB).

2. For each $x \in L$, it follows that

$$\begin{aligned} N_{I_{T^E,S^E,N^E}}(x) &= I_{T^E,S^E,N^E}(x, 0_L) \\ &= S^E(N^E(x), T^E(x, 0_L)) \\ &= S^E(N^E(x), 0_L) = N^E(x) \end{aligned}$$

□

In which follows, some results on properties preserved by the extension method via retractions and demonstrate proposition about the relationship of extension of QL-implications and its properties is presented.

QL-implications and exchange principle (EP)

Proposition 14 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Suppose T is a t-norm, S a t-conorm, and N a fuzzy negation defined on M respectively and $I_{T,S,N}$ is the QL-implication on M generated by T, S, and N. If $I_{T,S,N}$ satisfies (EP) then $I^E_{T,S,N}$ satisfies (EP).*

Proof For each $x, y, z \in L$, we have that

$$\begin{aligned} I^E_{T,S,N}(x, I^E_{T,S,N}(y,z)) &= s(I_{T,S,N}(r_1(x), r_1(I^E_{T,S,N}(y,z)))) \\ &= s(I_{T,S,N}(r_1(x), r_1(s(I_{T,S,N}(r_1(y), r_1(z)))))) \\ &= s(I_{T,S,N}(r_1(x), I_{T,S,N}(r_1(y), r_1(z)))) \\ &= s(I_{T,S,N}(r_1(y), I_{T,S,N}(r_1(x), r_1(z)))) \\ &= s(I_{T,S,N}(r_1(y), r_1(s(I_{T,S,N}(r_1(x), r_1(z)))))) \\ &= s(I_{T,S,N}(r_1(y), r_1(I^E_{T,S,N}(y,z)))) \\ &= I^E_{T,S,N}(y, I^E_{T,S,N}(x,z)) \end{aligned}$$

□

QL-implications and identity principle (IP)

Proposition 15 *Let $M \trianglelefteq L$ with respect to (r_1, r_2, s). Suppose T is a t-norm, S a t-conorm, and N a fuzzy negation defined on M respectively and $I_{T,S,N}$ is the QL-implication on M generated by T, S, and N. If $I_{T,S,N}$ satisfies (IP), then I_{T^E,S^E,N^E} satisfies (IP).*

Proof Notice that if $x = 1_L$, then

$$\begin{aligned} I_{T^E,S^E,N^E}(1_L, 1_L) &= S^E(N^E(1_L), T^E(1_L, 1_L)) \\ &= S^E(0_L, 1_L) = 1_L \end{aligned}$$

On the other hand, supposing $x \neq 1_L$, hence

$$\begin{aligned} I_{T^E,S^E,N^E}(x, 1_L) &= S^E(N^E(x), T^E(x,x)) \\ &= s(S(r_2(s(N(r_1(x))))), r_2(s(T(r_1(x), r_1(x)))))) \\ &= s(S(N(r_1(x)), T(r_1(x), r_1(x)))) \\ &= s(I_{T,S,N}(r_1(x), r_1(x))) \\ &= s(1_M) = 1_L \end{aligned}$$

□

Proposition 16 *Let $M < L$ with respect (r_1, s). If $I_{T,S,N}$ is a QL-implication on M satisfying (IP), then $T^E(x,x) \geqslant N^E_S \circ N^E(x)$ for all $x \in L$.*

Proof For each $x \in L$, we have

$$\begin{aligned} T^E(x,x) &= s(T(r_1(x), r_1(x))) \\ &\geqslant s(N_S(N(r_1(x)))) \\ &= s(N_S(r_1(s(N(r_1(x)))))) \\ &= N^E_S(N^E(x)) = N^E_S \circ N^E(x) \end{aligned}$$

□

QL-implications and identity principle (LEM)

Suppose S is a t-conorm on M and N is a fuzzy negation on M given by $N^E(x) = s(N(r_2(x)))$ which satisfy property (LEM). If $1_L \neq x \in L$, then

$$\begin{aligned} S^E(N^E(x), x) &= s(S(r(s(N(r_2(x)))), r_2(x))) \\ &= s(S(N(r_2(x)), r_2(x))) \\ &= s(r_2(x)) \geqslant 1_L \end{aligned}$$

Since 1_L is the top element of lattice L then $S^E(N^E(x), x) = 1_L$.

In case $x = 1_L$, it follows that $S^E(N^E(1_L), 1_L) = s(S(r(s(N(r_2(1_L)))), r_2(1_L))) = s(S(N(1_M), 1_M)) = s(1_M) = 1_L$. Therefore, we can state that

Proposition 17 *Let $M > L$ with respect (r_2, s). If S is a t-conorm on M and N is a fuzzy negation on M given by $N^E(x) = s(N(r_2(x)))$ which satisfy property (LEM), then $S^E(N^E(x), x) = 1_L$ for all $x \in L$.*

The table below shows a description about the properties that are preserved by extension method via retraction and those that are not preserved by EMR.

This results shows that this extension method is efficient if one wishes to obtain a minimal extension of the operator. However, some important properties regarding to implications are not preserved by this extension method. For instance ordering property (OP).

As we can in the Table 1, every property resulting from those $x \in L/M$ that not satisfies $s(r(x)) = x$ implies in some problem for the extension method. This problem can be solved if we consider a more powerful extension method (via e-operators, for short EMEP) as one can see in [9]. The results shown in [9] allow us to say that the extension method via retraction is better to obtain minimal extension whereas EMEP is more efficient in preserving properties.

Conclusions

We have investigated in this paper the behavior of extension method via retractions when applied for lattice-valued QL-implications. As occurred for other fuzzy operators (t-norms, t-conorms, and negations, see [2]), the results have shown some properties of this class of implication are not preserved by this method. For instance, it does not preserve property (NP) (see Proposition 3.6). It is

Table 1 Properties preserved and not preserved by the extension method

Property	Preserve	Not preserved
(LB)	X	
(RB)	X	
(CC1)	X	
(CC2)	X	
(CC3)	X	
(CC4)	X	
(NP)		X
(L-NP)	X	
(EP)	X	
(IP)	X	
(OP)		X
(IBL)		X
(CP)		X
(P)		X
(LEM)		X

desirable to propose an extension method more efficient in preserving the properties of extended operator.

For future works, we also wish to apply the extension method we have developed in [9] for QL-implications (actually, we want to study the extension of a more general class of implication operators and its properties) and make a comparison of the results.

Endnotes

[1]An element $a \in L$ is called a supremum (infimum) if it is the least (greatest) element in L satisfying property $a \geqslant x$ ($a \leqslant x$) for all $x \in L$.

[2]If f and g are functions on a lattice L, it is said that $f \leq g$ if and only if $f(x) \leq_L g(x)$ for all $x \in L$.

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Authors' contributions
All of the results presented here were proposed and proved by us. We have introduced the extension method in our previous works. In this paper, our main contribution was to provide a study about which properties of lattice-valued QL-implications are preserved by the application of the extension method via retractions. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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