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On clique-colouring of graphs with few P_4's

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Abstract Let $G=(V,E)$ be a graph with n vertices. A *clique-colouring* of a graph is a colouring of its vertices such that no maximal clique of size at least two is monocoloured. A k-clique-colouring is a clique-colouring that uses k colours. The clique-chromatic number of a graph G is the minimum k such that G has a k-clique-colouring.

In this paper we will use the primeval decomposition technique to find the clique-chromatic number and the clique-colouring of well known classes of graphs that in some local sense contain few P_4's. In particular we shall consider the classes of extended P_4-laden graphs, p-trees (graphs which contain exactly $n-3$ P_4's) and $(q,q-3)$-graphs, $q \geq 7$, such that no set of at most q vertices induces more that $q-3$ distincts P_4's. As corollary we shall derive the clique-chromatic number and the clique-colouring of the classes of cographs, P_4-reducible graphs, P_4-sparse graphs, extended P_4-reducible graphs, extended P_4-sparse graphs, P_4-extendible graphs, P_4-lite graphs, P_4-tidy graphs and P_4-laden graphs that are included in the class of extended P_4-laden graphs.

Keywords Graphs · Clique-colouring · Primeval decomposition · P_4-structure

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1 Introduction

In this paper we are concerned with the so called clique-colouring of a graph. To introduce this concept we need the following definitions. A subset K of G is a *clique* if every pair of distinct vertices of K are adjacent in G. A *clique* is maximal if it is not properly contained in any other clique of G. A *hypergraph* is a pair $\mathcal{H}=(V,\mathcal{E})$, where V is the set of vertices of $\mathcal{H}$ and $\mathcal{E}$ is a family of non-empty subsets of V called edges. A k*-colouring* of $\mathcal{H}$ is a mapping $c: V \to \{1,2,\ldots,k\}$ such that for all $e \in \mathcal{E}$ with $|e| \geq 2$ there exist $u, v \in e$ with $c(u) \neq c(v)$, that is, there is no monocoloured edge of size at least two. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest k such that $\mathcal{H}$ has a k-colouring.

We will consider special hypergraphs: hypergraphs arising from graphs. Given a graph $G=(V,E)$, the *clique-hypergraph* of G is the hypergraph $\mathcal{H}(G)=(V,\mathcal{E})$, whose vertices are the vertices of G and whose edges are the maximal cliques of G. A k-colouring of $\mathcal{H}(G)$ will also be called a k-clique-colouring of G and the *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ the *clique-chromatic number* of G. In other words: a *clique-colouring* of a graph is a colouring of its vertices such that no maximal clique of size at least two is monocoloured.

Clique-colouring is harder than ordinary vertex colouring: it is coNP-complete even to check whether a 2-clique-colouring is valid [7]. The complexity of 2-clique-colourability is investigated in [26] where they show it is NP-hard to decide whether a perfect graph is 2-clique-colourable even for those with clique-number 3. A valid 2-clique-colouring is not a good certificate, since we cannot verify it in polynomial time. In [27] it is proved that it is Σ_2^p-complete to check whether a graph is 2-clique-colourable, even for odd-hole-free graphs [15]. However quite general classes of graphs have been proved to be 2-clique colourable or 3-clique colourable. In [7] it was proved that $K_{1,3}$-free graphs are 2-clique colourable and that almost all perfect graphs are 3-clique colourable. In [29] it was shown that every planar graph is 3-clique colourable and in [26] it was proposed a polynomial algorithm to decide if a planar graph is 2-clique colourable. The clique-colourability of several other classes of graphs has been investigated in [1, 9, 10, 14, 19].

In this paper we consider some classes of graphs that have been characterized in terms of special properties of the unique primeval decomposition tree associated to each graph of the class. The primeval decomposition tree of any graph can be computed in linear time [8] and therefore it is the natural framework for finding polynomial time algorithms of many problems.

Using the primeval decomposition technique, we will determine the clique-chromatic number of graphs with few P_4's. In particular we shall consider the classes of extended P_4-laden [16], p-trees [3] (graphs which contain exactly $n-3$ P_4's) and $(q, q-3)$-graphs, $q \geq 7$, [2] such that no set of at most q vertices induces more that $q-3$ distincts P_4's. As corollary we shall derive the clique-chromatic number and the clique-colouring of the classes of cograph [11, 12], P_4-reducible graphs [22], P_4-sparse graphs [20, 24], extended P_4-reducible graphs [17], extended P_4-sparse graphs [17], P_4-extendible graphs [23], P_4-lite graphs [21], P_4-tidy graphs [18] and P_4-laden graphs [16] that are included in the class of extended P_4-laden graphs. Furthermore we will extend these results to more general classes obtained by substituting vertices of the above classes by homogeneous sets.

In Sect. 2 we give some definitions and preliminary results. In Sect. 3 we show that any graph that it is not a non-separable p-connected graph is 2-clique-colourable. In Sect. 4 we show that the class of p-trees is 2-clique-colourable. In Sect. 5 we find the clique-chromatic number and the clique-colouring of the remaining classes of graphs mentioned above. These results lead to polynomial time algorithms for finding the clique-colouring and the clique-chromatic number of the above classes.

2 Preliminaries

2.1 Basic notions

Throughout this paper let $G = (V, E)$ be a finite simple undirected graph and let $|V| = n$ and $|E| = m$. The *complement graph* of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$.

For a vertex $v \in V$ the neighbourhood of v in G is $N(v) = \{u | uv \in E\}$ and for $U \subset V$, $N(U)$ is the set of vertices in $V - U$ that are adjacent to at least one vertex of U. A *clique* of G is a set of pairwise adjacent vertices of G and a *stable set* is a set of pairwise non-adjacent vertices of G. Given a subset U of V, let $G[U]$ stand for the subgraph of G induced by U. Let P_n denote the chordless path on n vertices and $n-1$ edges. Let C_n denote the chordless cycle with n vertices. If $n \geq 4$ then C_n is called a *hole* and its complement an *antihole*. A graph is called a *complete graph* if every pair of distinct vertices is connected by an edge. A graph is called *split graph* if its vertex set can be partitioned in a clique K and a stable set S. A split graph is a *spider* if and only if $|K| = |S| \geq 2$ and there exists a bijection f between S and K such that for each $v \in S$, either $N(v) = \{f(v)\}$ (*thin legs*) or $N(v) = K - \{f(v)\}$ (*thick legs*). The simplest spider is a P_4. In a P_4 with vertices u, v, w, x and edges uv, vw, wx, the vertices v and w are called *midpoints* whereas the vertices u and x are called *endpoints*.

A *module* of G is a set of vertices M of V such that each vertex in $V - M$ is either adjacent to all vertices of M, or to none. The whole V and every singleton vertex are *trivial* modules. Whenever G has only trivial modules it is called a *prime* graph. A non-trivial module is also called an *homogeneous set*. We say that M is a *strong* module if for any other module A the intersection $M \cap A$ is empty or equals either M or A. For a non-trivial graph G, the family $\{M_1, M_2, \ldots, M_p\}$ of all maximal (proper) strong modules is a partition of $V(G)$. This partition is the *modular decomposition of* G. We will often identify the modules M_i with the induced subgraphs $G_i = G[M_i]$.

Whenever a graph G has a non-trivial maximal module M, in order to get some of its structural properties, it is useful to contract the module M to one representative vertex m obtaining a new graph H where $V(H) = V(G) - M \cup \{m\}$ and $E(H) = E(G/M) \cup \{ym | y \in N(M)\}$. The graph G' obtained from G by shrinking every maximal non-trivial module to a single vertex is called the *characteristic graph* of G. If G is a prime graph the $G' = G$.

If G is a prime graph we consider the *substitution operation* which consists of substituting any vertex x of G by any graph H in such a way that all the vertices of H have the same adjacencies of x in G. Therefore in the new graph G' obtained with this substitution operation $V(H)$ is a homogeneous set.

Let G and G' be two vertex disjoint graphs. We can define the *parallel composition* of G and G' as the graph $G \cup G'$ so that $V(G \cup G') = V(G) \cup V(G')$ and $E(G \cup G') = E(G) \cup E(G')$. The *serial composition* of G and G' is the graph $G + G'$ defined by $V(G + G') = V(G) \cup V(G')$ and $E(G + G') = E(G) \cup E(G') \cup \{vv'$ for each $v \in V(G)$ and $v' \in V(G')\}$.

2.2 p-Connectness and primeval decomposition

Following the terminology of Jamison and Olariu [25], a graph G is *p-connected* (or, more extensively, *P_4-connected*) if, for each partition V_1, V_2 of V into two non-empty sets, there exists a chordless path of four vertices P_4 which contains vertices from V_1 and V_2. Such P_4 is a *crossing* between V_1 and V_2. An equivalent characterization of p-connected graphs can be given in terms of *p-chains*, a natural analogue of paths in the context of usual connectedness of graphs. A *p-chain* connecting vertices u and v is a sequence of pairwise different vertices $(v_1, v_2, \dots, v_k)$ such that

1. $u = v_1$, $v = v_k$, and
2. $X_i = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ induces a P_4, for $i = 1, 2, \dots, k-3$.

By Babel and Olariu [5, 6] a graph is p-connected if and only if for every pair of vertices in the graph there exists a p-chain connecting them.

The *p-connected components* of a graph G are the maximal induced p-connected subgraphs. Vertices of G that do not belong to any p-connected component of G are termed *weak vertices*. A p-connected graph is called *separable* if its vertex set can be partitioned into two non-empty sets V_1 and V_2 in such a way that each crossing P_4 has its midpoints in V_1 and its endpoints in V_2. For separable p-connected graph the following theorem holds.

Theorem 1 [25] *A p-connected graph is separable if and only if its characteristic graph is a split graph.*

Separable p-connected components play a crucial role in the theory of p-connectedness and their introduction is justified by the following general theorem.

Theorem 2 [25] *For an arbitrary graph G exactly one of the following conditions is satisfied*:

1. *G is disconnected*;
2. *$\overline{G}$ is disconnected*;
3. *There is a unique proper separable p-connected component H of G with vertex partition (V_1, V_2) such that every vertex outside H is adjacent to all vertices in V_1 and to no vertex in V_2*;
4. *G is p-connected.*

This theorem implies a decomposition scheme for arbitrary graphs called *primeval decomposition*.

For disconnected G, the maximal strong modules are the connected components. In this case $G = G_1 \cup G_2 \cup \dots \cup G_p$ is called *parallel*.

If $\overline{G}$ is disconnected, the maximal strong modules of G are the connected components of $\overline{G}$. In this case $G = G_1 + G_2 + \dots + G_p$ is called *serial*.

If both G and $\overline{G}$ are connected, then either G can be decomposed according to condition 3 of Theorem 2 and in this case G is called a *decomposable neighbourhood graph* or G is a p-connected graph that may be either separable or non-separable.

By repeating the process we can associate to any non-empty graph G its unique *primeval decomposition tree* $T(G)$. The root of $T(G)$ is G, the leaves are the p-connected components and the weak vertices of G and the internal nodes of $T(G)$ are labeled with P, S or N (for parallel, serial, or decomposable neighbourhood graph, respectively).

3 The clique-colouring of graphs which are not non-separable *p*-connected graphs

In this section we will show that the clique-chromatic number of any graph which is not a non-separable p-connected graph is equal to 2. In the next sections we shall consider classes of graphs that contain non-separable p-connected graphs of special type.

From now on we will assume that the graphs we are considering are connected, otherwise we could always consider separately each connected component.

First we show that in order to find an optimal clique-colouration of a graph, it is enough to have an optimal clique-colouration of its characteristic graph.

Theorem 3 *Every graph G has the same clique-chromatic number of its characteristic graph G'. Given an optimal clique-colouration of G', an optimal clique-colouration of G is obtained assigning to every vertex belonging to a maximal strong module of G the same colour of its representative vertex in G'.*

Proof Let G be a graph and let G' be its characteristic graph. Let $\{M_1, M_2, \dots, M_p\}$ be the family of all maximal (proper) strong modules of G and let $\{v_1, v_2, \dots, v_p\}$ be the set of the corresponding characteristic vertices in G'. Let us assume that an optimal clique-colouration of G' is known. Let $K' = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$ be any maximal clique of G'. By replacing each vertex v_{i_j} of every K' with any maximal clique of the corresponding module M_{i_j} we obtain all the maximal cliques of G. In fact each set of vertices generated is a clique since each pair of vertices are adjacent either by construction or by definition of module. The maximality follows by the maximality of each clique K' of G' and the maximality of the cliques replacing each vertex of K'. Now, we can extend the clique colouration of G' to G by assigning the colour of each representative vertex of G' to every vertex of the corresponding homogeneous set in G. No maximal clique of size at least 2 is monocoloured since every maximal clique of G contains an induced subgraph isomorphic to a maximal clique of G' that is not monocoloured by hypothesis. The optimality of the colouration of G follows from the optimality of the colouration of G'. □

Theorem 4 *Every serial graph is* 2*-clique colourable.*

Proof Let $G = G_1 + G_2 + \cdots + G_p$, $p \geq 2$, be a serial graph. The characteristic graph of G is a complete graph that is 2-clique colourable. Therefore G is also 2-clique-colourable by Theorem 3. In fact, it is sufficient to colour the vertices of G_1 with colour 1 and the remaining vertices with colour 2. □

Lemma 1 *Every split graph is* 2*-clique colourable.*

Proof Let G be a split graph. Then its vertex set can be partitioned into a maximal clique K and an independent set S. Every maximal clique of G is either K or a subset of K with at most one vertex of S. Choose any vertex of K, say x, and colour it with colour 1. Colour the remaining vertices of K with colour 2. Colour the vertices of $N_S(x)$ with colour 2 and the remaining vertices of S with colour 1. We claim that this colouring is a 2-clique-colouring of G. In fact, if K is a maximal clique of G it has two different colours. Any other maximal clique of G contains vertices of K and only one vertex of S. If it contains x then it is 2-clique-colourable since all its adjacent vertices have colour 2. Else, if it contains only vertices of K coloured with colour 2, then the vertex of S by construction has colour 1. So it is 2-clique colourable. □

Theorem 5 *Every separable p-connected graph is* 2*-clique-colourable.*

Proof The characteristic graph of every separable p-connected graph G is a split graph by Theorem 1. Any split graph is 2-clique-colourable by Lemma 1. Therefore G is also 2-clique-colourable by Theorem 3. □

Theorem 6 *Every decomposable neighbourhood graph is* 2*-clique colourable.*

Proof Let $G = (V, E)$ be a decomposable neighbourhood graph. Then by condition 3 of Theorem 2 there is a unique proper separable p-connected component H of G with vertex partition (V_1, V_2) such that every vertex in $V_3 = V \setminus (V_1 \cup V_2)$ is adjacent to all vertices in V_1 and to no vertex in V_2. Then V_3 is a module in G. The characteristic graph of G is obtained by shrinking V_3 to a single vertex, and by substituting H with its characteristic graph, which is a split graph by Theorem 1. Then the characteristic graph of G is also a split graph. Therefore G is 2-clique colourable by Lemma 1 and Theorem 3. □

4 The clique-colouring of p-trees and p-forests

The purpose of this section is to show that a special subclass of the P_4-connected graphs, called p-trees is 2-clique-colourable. This class, introduced by Babel in [3], is provided with structural properties that can be expressed in a quite analogous way to the characterization of ordinary trees. A vertex is called a *p-end-vertex* if it belongs to exactly one P_4. A *p-cycle* is a p-connected graph with no p-end-vertices, and which is minimal with this property. The class of *p-trees* is the class of p-connected graphs without induced p-cycles and containing exactly $n-3$ P_4's. A *p-forest* is a graph whose p-connected components are p-trees [3].

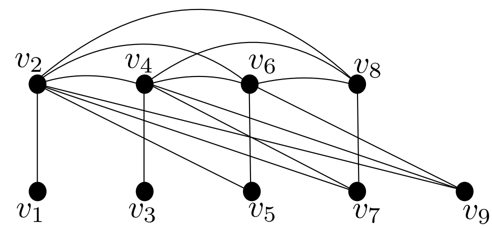

Fig. 1 The graph Q_9

In the following we shall use the characterization of p-trees given in [4], based on the structure of the p-chains in a p-tree, which we recall for reader's convenience.

A p-chain $X = (v_1, v_2, \ldots, v_k)$ is *simple* if the only P_4's contained in $G[\{v_1, v_2, \ldots, v_k\}]$ are induced by the set of vertices $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for $i = 1, 2, \ldots, k-3$. In other words a p-chain X is simple if and only if the vertices of the p-chain induce precisely $k-3$ P_4's.

Let P_k be a path with vertex set $V = \{v_1, v_2, \ldots, v_k\}$ and edge set $E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ and let Q_k be a split graph with vertex set $V = \{v_1, v_2, \ldots, v_k\}$, where $A = \{v_{2i-1} \in V | 1 \leq 2i-1 \leq k\}$ is a stable set, $B = \{v_{2i} \in V | 2 \leq 2i \leq k\}$ is a clique and the edges connecting each vertex of B to the vertices of A are $\{v_{2i}v_{2i-1}$ and $v_{2i}v_{2j+1}, j > i\}$ (see Fig. 1). The ordered sequence $(v_1, v_2, \ldots, v_k)$ of the vertices of P_k $(k \geq 4)$, Q_k $(k \geq 5)$ and their complements are simple p-chains. It has been proved that graphs whose vertices can be ordered in a simple p-chains are p-trees and it turns out that every p-tree can be generated starting from a simple p-chain extended by a number of p-end-vertices which can eventually be replaced by cographs [4].

A *spiked p-chain P_k* is a path P_k, $k \geq 5$, eventually extended, by introducing two additional vertices x and y such that x is adjacent to v_2 and v_3 and y is adjacent to v_{k-1} and v_{k-2}; moreover we request that x and y do not belong to a common P_4 (see Fig. 2).

A *spiked p-chain Q_k* is a split graph Q_k, $k \geq 6$, with eventually, additional vertices $z_2, z_3, \ldots, z_{k-5}$ such that

$$N(z_i) = \{v_2, v_4, \ldots, v_{i-1}, v_{i+1}\} \cup \{z_2, z_4, \ldots, z_{i-1}\} \quad \text{for } i \text{ odd};$$

$$\overline{N}(z_i) = \{v_1, v_3, \ldots, v_{i-1}, v_{i+1}\} \cup \{z_3, z_5, \ldots, z_{i-1}\} \quad \text{for } i \text{ even}.$$

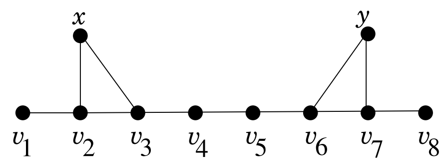

Fig. 2 The spiked p-chain P_8

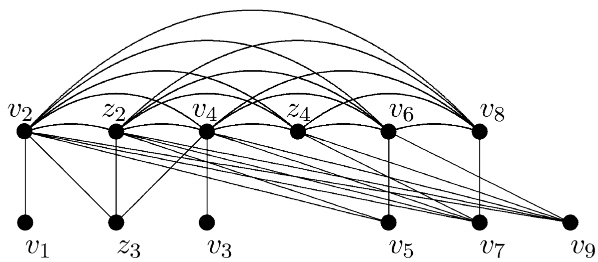

Fig. 3 The spiked p-chain Q_9

A spiked p-chain is shown in Fig. 3. A spiked p-chain $\overline{P}_k$ (or $\overline{Q}_k$) is the complement of a spiked p-chain P_k (or Q_k).

It is easy to verify that v_1, x, y, v_k and $v_1, z_2, z_3, \ldots, z_{k-5}, v_k$ are the only p-end-vertices of spiked P_k ($\overline{P}_k$) and Q_k ($\overline{Q}_k$) respectively.

Finally we have the following characterization of p-trees given by Babel.

Theorem 7 [4] *A graph is a p-tree if and only if it is either a P_4 with one vertex replaced by a cograph or a spiked p-chain with the p-end-vertices replaced by cographs.*

Theorem 8 *Every p-tree is 2-clique-colourable.*

Proof From Theorem 7, it follows that the only homogeneous sets of a p-tree are the cographs eventually replacing the p-end-vertices of G. By Theorem 3 graph G has the same clique-chromatic number of its characteristic graph. Therefore it is enough to consider the clique-colouration of the spiked p-chains P_k, Q_k and their complements.

Notice that a prime p-tree is separable if and only if it is either a spiked p-chain Q_k or $\overline{Q}_k$, ($k \geq 6$) or a P_4. If G is a separable p-tree then it is 2-clique-colourable by Theorem 5. If G is not separable then G is a spiked p-chains P_k or $\overline{P}_k$, ($k \geq 5$). Let now consider the case when G is a spiked p-chains P_k. If the p-end-vertices x and y are not present then G is a path P_k that can be 2-clique-coloured by assigning to the vertices of the path alternatively colours 1 and 2. If the p-end-vertices x or y are present they belong to a maximal clique of G that already contains a 2-coloured edge so they can be coloured with colour 1 or 2. Let G be a spiked p-chain $\overline{P}_k$. If the p-end-vertices x and y are not present then G is a $\overline{P}_k$. Let $v_1, v_2, \ldots, v_k$ be the sequence of the vertices of P_k. The vertex set of a maximal clique of $\overline{P}_k$ is any maximal ordered sequence of vertices $S = v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ such that for each two consecutive vertices v_{i_p} and v_{i_q}, $i_q - i_p$ is either 2 or 3. Since S is a maximal sequence then v_{i_1} must belong to the set $\{v_1, v_2\}$ and v_{i_s} to the set $\{v_{k-1}, v_k\}$. Then every maximal clique of $\overline{P}_k$ contains one of the edges v_1v_k or v_1v_{k-1} or v_2v_k or v_2v_{k-1}. Therefore, in order to obtain a 2-clique colouration of $\overline{P}_k$ it is enough to colour the vertices v_1 and v_2 with colour 1, the vertices v_k and v_{k-1} with colour 2 and any other vertex with colour 1 or 2. If the p-end-vertices x or y are present they belong to a maximal clique of G that already contains a 2-coloured edge so they can be coloured with colour 1 or 2. □

Theorem 9 *Every p-forest is 2-clique-colourable.*

Proof A p-forest G is either a serial graph or a decomposable neighbourhood graph or a p-tree. Then G is 2-clique-colourable by Theorems 4, 6 and 8 respectively. □

5 The clique-colouring of graphs with few P_4's

In this section we shall consider graphs which, in a local sense, contain only a restricted number of P_4's. It all started with the class of *cographs*, which is a class of graphs where no P_4 is allowed to exist. These graphs have been investigated in [11, 12] and many nice structural results are known. In particular it has been shown in [11] that every connected cograph is a serial graph and therefore it is 2-clique-colourable by Theorem 4.

The study of cographs has been extended by many authors. Hoáng [20] introduced the class of P_4-sparse graphs, which is the class such that no set of five vertices induces more than one P_4. The non-trivial leaves of its associated primeval tree are spiders [24]. Jamison and Olariu [21–23] introduced the class of P_4-reducible graphs, P_4-extendible and P_4-lite. The P_4-reducible graphs are the graphs such that no vertex belongs to more than one P_4, and its p-connected components are P_4's. The P_4-extendible are graphs where each p-connected component consists of at most five vertices. Each p-connected component is either P_5 or $\overline{P_5}$ or C_5, or P_4 with one vertex eventually substituted by a homogeneous set with cardinality two. The P_4-lite are graphs such that every induced subgraph with at most six vertices either contains at most two P_4's or is isomorphic to a spider. The p-connected components of a P_4-lite graph are either a spider (possibly with one vertex replaced by a homogeneous set of cardinality 2) or one of the graphs P_5, $\overline{P_5}$. The P_4-laden [16] are graphs such that every induced subgraph with at most six vertices either contains at most two P_4's or it is isomorphic to a split graph. The p-connected components of a P_4-laden graph are spiders (possibly with one vertex replaced by a homogeneous set of cardinality 2), split graphs, P_5's or $\overline{P_5}$'s [16]. The above classes are ordered as follows:

cographs $\subset$ P_4-reducible $\subset$ P_4-sparse $\subset$ P_4-lite $\subset$ P_4-laden.

Another generalization of the above classes are the extended P_4-reducible [17], extended P_4-sparse graphs [17], P_4-tidy graph [18] and the extended P_4-laden graphs [16]. These classes are obtained from P_4-reducible and P_4-sparse graphs, P_4-lite and P_4-laden respectively, by also allowing C_5's as p-connected components. All the previously mentioned classes are included in the class of extended P_4-laden graphs.

Using the primeval decomposition we have the following.

Theorem 10 *Every extended-P_4-laden graph G is 2-clique-colourable with the exception of C_5 which is 3-clique-colourable.*

Proof If G is a serial graph or a decomposable neighbourhood graph, it is 2-clique-colourable by Theorem 4 and Theorem 6 respectively. If G is a p-connected graph different from C_5, then either it is isomorphic to a P_5 or $\overline{P_5}$ that are trivially 2-clique-colourable or it is a separable p-connected graph isomorphic either to a spider graph (possibly with one vertex replaced by a homogeneous set of cardinality 2), or to a split graph. The characteristic graph in both cases is a split graph and therefore it is 2-clique-colourable by Lemma 1. If G is a C_5 then G is 3-clique-colourable. □

Corollary 1 *Every cograph, P_4-reducible, P_4-sparse and P_4-lite graphs are 2-clique-colourable.*

Corollary 2 *Every P_4-extendible, extended P_4-reducible and extended P_4-sparse graphs are 2-clique-colourable with the exception of C_5 which is 3-clique-colourable.*

A generalization of some of the above classes was given by Babel and Olariu. They proposed to call a graph a (q,t)-graph if no set of at most q vertices induces more than t distinct P_4's. In this terminology, the cographs are precisely the $(4,0)$-graphs, the P_4-sparse graphs coincide with the $(5,1)$-graphs. In particular the $(7,4)$-graphs properly contain all cographs, P_4-reducible, P_4-sparse, p-trees and p-forests. The $(9,6)$-graphs additionally contain all P_4-extendible, extended P_4-reducible and extended P_4-sparse graphs.

The p-connected $(q,q-3)$-graph has been characterized as follows.

Theorem 11 [4] *Let G be a p-connected graph with n vertices.*

(i) *If $n \geq 7$, then G is a $(n,n-3)$ graph if and only if G is a p-tree.*

(ii) *If $n > q$, $q \in \{7,9\}$, then G is a $(q,q-3)$-graph if and only if precisely one of the following conditions holds*
  (a) *G is a p-tree*;
  (b) *G is a hole or antihole*;
  (c) *G is a spider.*

(iii) *If $n > q$, $q = 8$, or $q \geq 10$, then G is a $(q,q-3)$-graph if and only if precisely one of the following conditions holds*
  (a) *G is a p-tree*;
  (b) *G is a hole or antihole.*

In order to find the clique-colouration of $(q,q-3)$-graphs we will need the following result.

Theorem 12 *Every antihole $\overline{C}_k$, with $k > 5$ is 2-clique-colourable.*

Proof Let $v_1, v_2, \ldots, v_k$, $k \geq 6$, be the sequence of the vertices of C_k. Any 2-colouring where v_1, v_2, v_3 have colour 1 and v_4, v_5, v_6 have colour 2 is a 2-clique-colouring. In fact a clique that misses 3 consecutive vertices v_i, v_{i+1}, v_{i+2} is not maximal because we could include v_{i+1} obtaining a bigger clique. Hence, any clique has (at least) one vertex in $\{v_1, v_2, v_3\}$ and (at least) one vertex in $\{v_4, v_5, v_6\}$ and it is 2-clique-coloured. □

Note that Theorem 12 can be obtained also as a particular case of results contained in [1, 7].

Now we are ready to prove the following theorem.

Theorem 13 *Let $q \geq 7$ be a fixed integer and G a $(q,q-3)$-graph with n vertices, $n \geq q$. G is 2-clique-colourable, unless it is a C_n with n odd (which is 3-clique-colourable).*

Proof If $n = q$ then G is a p-tree, which is 2-clique-colourable by Theorem 8. If $n > q$ then G is either a serial graph or a decomposable neighbourhood graph or a p-connected graph. In the first two cases it is 2-clique-colourable by Theorems 4 and 6 respectively. In the last case, G has more than 7 vertices and it is either a p-tree, or a hole, or an antihole, or a spider by Theorem 11. Every p-tree is 2-clique-colourable by Theorem 8. Every spider is a split graph and therefore it is 2-clique-colourable by Lemma 1. Every antihole is 2-clique-colourable by Theorem 12. Finally every hole is 2-clique-colourable if it is even, otherwise is 3-clique-colourable. □

6 Final remarks

We would like to point out that, in order to study the clique-colouration of a graph, it is enough to consider the clique-colouration of its characteristic graph that can be obtained in linear time through the modular decomposition of that graph [13, 28]. We have showed that every serial graph, every separable p-connected graph and every neighbourhood decomposable graph is 2-clique-colourable. Furthermore we have solved the clique-colouring problem for classes of graphs that contain very special non-separable p-connected graphs. Moreover, if the clique-colouration of a prime graph G is known, then any graph obtained by substituting any vertex of G with any graph H has the same clique-colouration of G. The problem remains open for general non-separable p-connected graphs.

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