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Complexity separating classes for edge-colouring and total-colouring

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Abstract The class of unichord-free graphs was recently investigated in a series of papers (Machado et al. in Theor. Comput. Sci. 411:1221–1234, 2010; Machado, de Figueiredo in Discrete Appl. Math. 159:1851–1864, 2011; Trotignon, Vušković in J. Graph Theory 63:31–67, 2010) and proved to be useful with respect to the study of the complexity of colouring problems. In particular, several surprising complexity dichotomies could be found in subclasses of unichord-free graphs. We discuss such results based on the concept of "separating class" and we describe the class of bipartite unichord-free as a final missing separating class with respect to edge-colouring and total-colouring problems, by proving that total-colouring bipartite unichord-free graphs is NP-complete.

Keywords Theoretical computer science · Computational complexity · Colouring of graphs · Total chromatic number · Bipartite unichord-free

1 Complexity dichotomies and separating classes

Given a class $\mathbb{G}$ of graphs and a graph (decision) problem φ belonging to NP, we say that a *full complexity dichotomy* of $\mathbb{G}$ was obtained if one describes a partition of $\mathbb{G}$ into $\mathbb{G}_1, \mathbb{G}_2, \ldots$ such that φ is classified as polynomial or NP-complete when restricted to each $\mathbb{G}_i$. The concept of *full complexity dichotomy* is particularly interesting for the investigation of NP-complete problems: as we partition a class $\mathbb{G}$ into NP-complete subclasses and polynomial subclasses, it becomes clearer why the problem is NP-complete in $\mathbb{G}$. Clearly, if a problem φ is polynomial in $\mathbb{G}$, any partition of $\mathbb{G}$ will determine polynomial subclasses; and if a problem is NP-complete in $\mathbb{G}$, any *finite* partition $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_n$ of $\mathbb{G}$ will determine *at least one* subclass $\mathbb{G}_i$ such that φ restricted to $\mathbb{G}_i$ is NP-complete or the recognition of $\mathbb{G}_i$ is NP-complete.

Another interesting tool for the analysis of problems is the idea of separating class. A class $\mathbb{G}$ of graphs is a *separating class* for problems φ_1 and φ_2 if φ_1 is NP-complete when restricted to $\mathbb{G}$ and φ_2 is polynomial when restricted to $\mathbb{G}$—or vice versa. The nice idea about a separating class is that it shows how the same structure may define a polynomial problem and an NP-complete problem. We recall that the "dual" idea of *separating problem*—a problem that has distinct complexities when restricted to distinct classes—was already investigated in [3]. For instance, vertex-colouring is a separating problem for planar graphs (NP-complete) and its subclass of series-parallel graphs (polynomial) [3]. It is surprising that for most classes proposed in [3] the complexity of edge-colouring remains challenging open.

Recall that an edge-colouring is an association of colours to the edges of a graph in such a way that adjacent edges receive different colours, and a total-colouring is an association of colours to the vertices and edges of a graph in such a way that adjacent or incident elements receive different colours. A graph G is said to be *Class 1* if it admits an edge-colouring using a number of colours equal to its maximum degree $\Delta(G)$; graph G is said to be *Type 1* if it admits a total-colouring using $\Delta(G)+1$ colours. The study of the complexity of edge-colouring and total-colouring is challenging in the sense that both problems are NP-complete, the restrictions to very few classes are known to be polynomial, and

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the Total-Colouring Conjecture[1] is a central open problem in Graph Theory. In the present work, we consider separating classes for edge-colouring and total-colouring. We claim, however, that not any separating class for these problems is of interest. Observe that it is quite easy to construct ("artificial") separating classes for these problems. Recall that both edge-colouring and total-colouring are NP-complete problems and consider the classes

$$\mathbb{G}_x := \{G : \Delta(G) = x \text{ and } \omega(G) = \Delta(G) + 1\}$$

for $x \in \{3, 4, 5, \ldots\}$ (recall that $\Delta(G)$ denotes the maximum degree in G and $\omega(G)$ denotes the size of a maximum clique in G).

We invite the reader to check that each $\mathbb{G}_x$ with even x is a separating class where edge-colouring is polynomial and total-colouring is NP-complete. On the other hand, each $\mathbb{G}_x$ with odd x is a separating class where edge-colouring is NP-complete and total-colouring is polynomial.

We argue, however, that classes like $\mathbb{G}_x$ are not "interesting" classes, in the sense that the polynomiality of the problems edge-colouring or total-colouring do not arise from any nice structural property, but simply from the fact that a "large clique" forces a NO answer in any case. Such situation is analogous to the one that led to the definition of a perfect graph: recall that a graph G is perfect if *every induced subgraph* G' of G satisfies $\chi(G') = \omega(G')$ (recall that $\chi(G')$ denotes the chromatic number of G'). This avoids the occurrence of uninteresting graph classes such as the ones that contain "large cliques". In this sense, we argue that the separating classes that should be considered in the context of colouring problems should be *closed* under taking induced subgraphs.

Colouring problems in unichord-free graph classes A graph is unichord-free if it does not contain a cycle with a unique chord (as an induced subgraph). Unichord-free graphs recently attracted great interest because of their rich structure that led to interesting—even surprising—results regarding the complexity of colouring problems. The class of unichord-free graphs is closed under taking induced subraphs—hence interesting to be considered in the context of separating classes. We recall the main results on colouring unichord-free graphs:

- Every unichord-free graph G is $\max\{3, \omega(G)\}$-vertex-colourable [11].
- Edge-colouring and total-colouring restricted to unichord-free graphs are NP-complete problems [7, 8].
- Edge-colouring restricted to square-free unichord-free graphs with maximum degree 3 is NP-complete [8]; every non-complete square-free unichord-free graph with maximum degree at least 4 is Class 1 [8].
- Every non-complete square-free unichord-free graph with maximum degree at least 3 is Type 1 [6, 7].
- Every chordless[2] graph G with $\Delta(G) \geq 3$ is Class 1 and Type 1 [9].

The third observation determines that a full complexity dichotomy of square-free unichord-free graphs is described for edge-colouring. The partition is constructed as a function of the maximum degree, and the complexity dichotomy is particularly surprising, so far unmatched in the literature: exactly one part is NP-complete, namely the part of unichord-free graphs with maximum degree 3. In each of the other parts, the problem is polynomial.

Observe, additionally, that the class of square-free unichord-free graphs is a separating class for edge-colouring and total-colouring problems. While the class is not the first separating class for edge-colouring and total-colouring described in the literature (bipartite graphs is one such example), it is the first one where edge-colouring is actually "harder" than total-colouring. The unexpectability of such result comes from the fact that total-colouring is traditionally viewed as a problem "harder" than edge-colouring. The NP-completeness proof of total-colouring [10] is a reduction from edge-colouring, and most classes investigated in the context of total-colouring are classes where edge-colouring is well understood [1, 2, 12].

The above cited results on edge-colouring and total-colouring in subclasses of unichord-free graphs motivate the search for a subclass where edge-colouring is polynomial and total-colouring is NP-complete. We identify bipartite unichord-free graphs as an example of such separating class. We describe one additional motivation for the class: it is known that total-colouring is NP-complete for bipartite graphs and for unichord-free graphs, hence it is natural to consider the intersection of the two classes.

In Sect. 2 we determine the NP-completeness of total-colouring bipartite unichord-free graphs establishing this class as a separating class for edge-colouring and total-colouring problems (recall that bipartite graphs are Class 1). Figure 1 presents complexity results on edge-colouring and total-colouring in several subclasses of unichord-free graphs, considering the results in the literature and the result of the present paper.

2 Bipartite unichord-free graphs as a separating class for edge-colouring and total-colouring

Our proposed reduction from the NP-complete problem of edge-colouring a regular graph [5] to total-colouring a regular bipartite unichord-free graph follows the strategy of [10],

[1] The TCC states that every graph G is $(\Delta(G) + 2)$-total-colourable.

[2] Graph such that every cycle is induced—hence, a subclass of unichord-free.

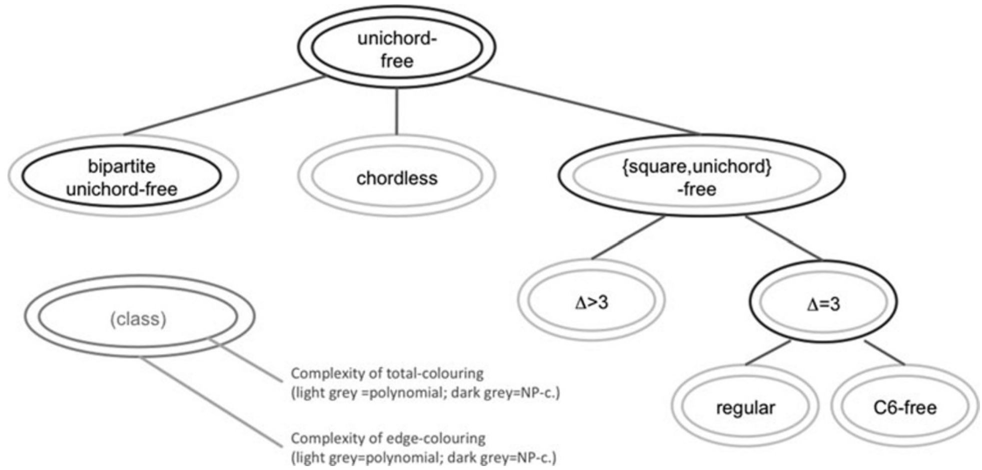

Fig. 1 Complexity of edge-colouring and total-colouring in subclasses of unichord-free

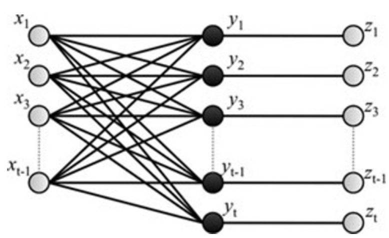

Fig. 2 Graph S_t: in any optimal total-colouring all pendant edges receive same colour

but modifies gadgets from [7, 10] to obtain a constructed graph in the target class of regular bipartite unichord-free graphs.

Gadget S_t, $t \geq 3$ in Fig. 2 is obtained from the complete bipartite graph $K_{t-1,t}$ by the addition of t pendant edges incident to the t vertices with degree $t-1$ [10]. The bipartite gadget H_t, $t \geq 3$, is constructed by taking three copies of S_t and identifying two pairs of pendant edges (H_t is depicted in Fig. 3). Observe that H_t has $3t-4$ pendant vertices, $2t-2$ of which belong to the "black" partition and $t-2$ of which are belong to the "white" partition.

The bipartite graph R_t, $t \geq 3$, is constructed by taking $t+1$ copies of H_t and identifying pairs of pendant vertices of different copies of H_t (the first two graphs of the family are depicted in Fig. 4). Gadget R_3 is defined in [10] and extended to family R_t in [7]. Each copy of H_t has one of its black pendant vertices identified with one black pendant vertex of each of the other copies. Constructing R_t in this fashion, we reach a situation where each copy of H_t still has $t-2 \geq 1$ pendant "white vertices" and $t-2 \geq 1$ pendant "black vertices".

The final bipartite gadget $F_{n,t}$ in Fig. 5 is constructed by taking n copies of H_t and connecting them through the identification of pendant vertices in a linear fashion [10].

Lemma 1 (McDiarmid e Sánchez-Arroyo [10]) *Consider graph* S_t, $t \geq 3$.

1. *There exists a* $(t+1)$*-total-colouring of* S_t *such that each of the vertices* $y_1, y_2, \ldots, y_t$ *has a different colour.*
2. *Any* $(t+1)$*-total-colouring of* S_t *colours all pendant edges the same.*

Following the strategy of [10], the proofs of Lemmas 2, 3, and 4 can be obtained by applying similar reasoning to the present modified corresponding gadgets.

Lemma 2 *Consider graph* H_t.

1. *Consider a partial* $(t+1)$*-total-colouring* π' *of* H_t *such that the pendant edges are coloured the same and the pendant vertices are coloured. This* $(t+1)$*-total-colouring extends to a* $(t+1)$*-total-colouring of* H_t.
2. *Any* $(t+1)$*-total-colouring of* H_t *colours all pendant edges the same.*

Lemma 3 *Consider graph* R_t, $t \geq 3$.

1. *Consider a partial* $(t+1)$*-total-colouring of* R_t *in which* $t+1$ *pendant edges attached to* $t+1$ *different copies of* H_t *have different colours and the corresponding pendant vertices are coloured. This colouring extends to a* $(t+1)$*-total-colouring of* R_t.
2. *Any* $(t+1)$*-total-colouring of* R_t *colours all pendant edges of each copy of* H_t *with a unique colour different of the colour of the other pendant edges.*

Lemma 4 *Consider graph* $F = F_{n,t}$, $n \geq 2$ *and* $t \geq 3$.

1. *Consider a partial* $(t+1)$*-total-colouring of* F *in which each pendant edge has the same colour and each pendant vertex is coloured. This colouring extends to a* $(t+1)$*-total-colouring of* F.
2. *Any* $(t+1)$*-total-colouring of* F *colours all pendant edges the same.*

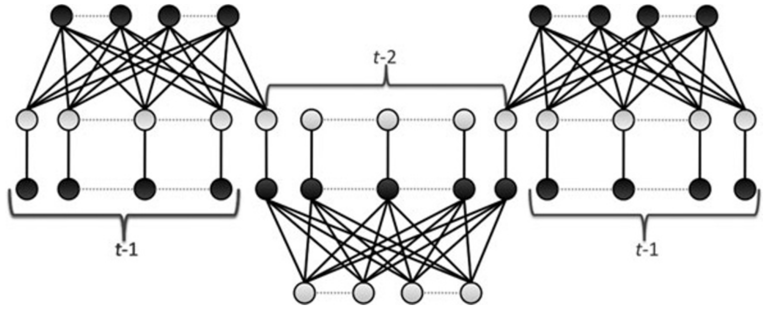

Fig. 3 Graph H_t: in any optimal total-colouring all pendant edges receive same colour

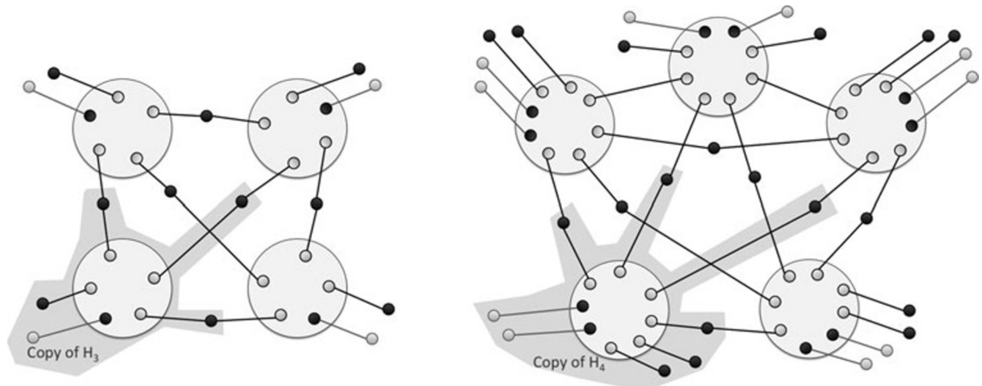

Fig. 4 "Replacement graphs" R_3 and R_4

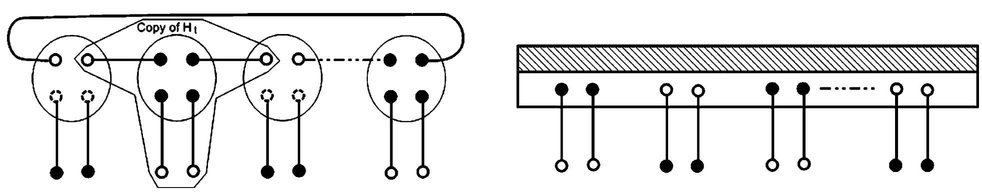

Fig. 5 Graph $F_{n,t}$

Theorem 1 *For each* $\Delta \geq 3$, *the total-colouring problem restricted to bipartite unichord-free graphs with maximum degree* Δ *is NP-complete.*

Proof Recall that total-colouring is in NP. Let G be a Δ-regular graph. We construct a bipartite unichord-free graph G' with maximum degree Δ such that G' is $(\Delta+1)$-total-colourable if and only if G is Δ-edge-colourable.

Step 1. Construct an intermediate graph $\tilde{G}$ substituting each vertex v of G for a copy of R_Δ (name this copy $R_\Delta^{(v)}$), associating Δ of the pendant edges of R_Δ—from distinct copies of H_t—with the Δ edges of G incident to v. Notice that R_Δ has $\Delta+1$ internal copies of H_Δ, so that one of these copies provides no pendant edge to be associated with other edges; such copy is said to be *unused*. The pendant edges can be chosen (among those incident to black or white pendant vertices) in such a way that $\tilde{G}$ is bipartite.

Step 2. Construct G' by taking in $\tilde{G}$, for each copy of R_Δ, one pendant edge from the unused internal copy of H_Δ, and by identifying this edge with a pendant edge from $F_{n,\Delta}$ (any $n \geq |V(G)|$ allows the identification of all pendant edges of $\tilde{G}$). Once again, the edges can be chosen (among those incident to black or white vertices) in such a way that the graph stays bipartite.

We invite the reader to check that the constructed graph G' is always bipartite and unichord-free. Figure 6 shows an example of construction of a bipartite unichord-free graph G' of maximum degree 3 from a cubic graph G.

Notice that each edge $uv \in E(G)$ is mapped to an edge $(uv)' \in E(G')$ with endvertices in $R_\Delta^{(u)}$ and $R_\Delta^{(v)}$.

Suppose that G' is $\Delta+1$-total-colourable and consider a $(\Delta+1)$-total-colouring of G'. By construction, the two edges $(uv)'$ and $(uw)'$ associated to edges uv and uw of G are incident to the same copy $R_\Delta^{(u)}$—by part 2 of Lemma 3 these edges $(uv)'$ and $(uw)'$ receive different colours. Moreover, each copy of R_Δ has one edge that is incident to $F_{n,t}$. So, the unique colour used in part 2 of Lemma 4 in the pendant edges of $F_{n,t}$ does not appear in the edges of G' to which the edges of G are mapped—and only Δ colours are used in these edges. Hence, a $(\Delta+1)$-total-colouring of G' determines a Δ-edge-colouring of G given by the colours of the edges of G' that are associated to the edges of G.

Now assume that G is Δ-edge-colourable and consider a Δ-edge-colouring of G. Give the colour of each edge $uv \in$

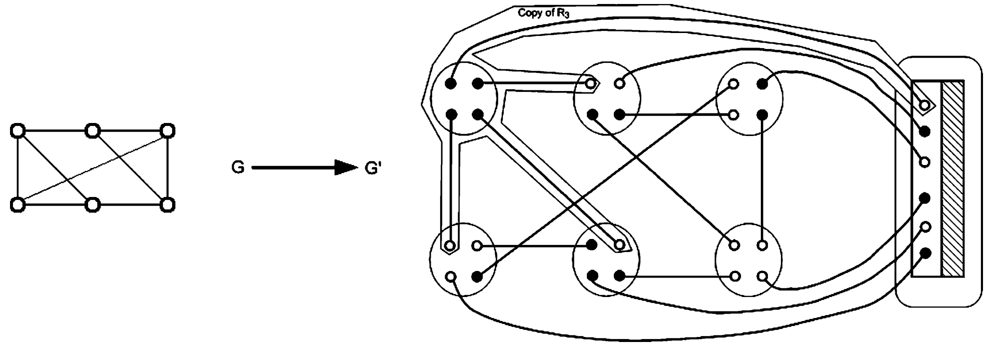

Fig. 6 Example of construction of G' from G

$E(G)$ to its corresponding edge $(uv)' \in E(G')$. Additionally give colour $\Delta + 1$ to the edges of G' that are incident to one copy of R_Δ and $F_{n,t}$. By part 1 of Lemmas 3 and 4, this partial $(\Delta + 1)$-total-colouring of G' extends to a $(\Delta + 1)$-total-colouring of G' by colouring the internal elements of each copy of R_Δ and $F_{n,t}$. □

Although the result achieved in Theorem 1 is strong when compared with previous classes [7, 10], we should point out that the modified simpler gadgets provide a straightforward proof when compared to [10] and the technique of induction on the minimum degree described in [7] can still be applied to obtain a regular graph. So, the total-colouring problem is NP-complete when restricted to *regular* bipartite unichord-free graphs.

3 Final remarks

The present work addresses the complexity of edge-colouring and total-colouring restricted to subclasses of unichord-free graphs. In particular, we determine the NP-completeness of total-colouring restricted to bipartite unichord-free graphs—a class polynomially Δ-edge-colourable. The importance of the result is evident considering the previous NP-completeness results of total-colouring for both bipartite graphs and unichord-free graphs. Moreover, it is interesting to note that unichord-free graphs have a subclass for which edge-colouring is NP-complete but total-colouring is polynomial, namely, the class of {square,unichord}-free graphs [7, 8]. Hence, it is of great interest to determine a subclass of unichord-free graphs for which the complexities of edge-colouring and total-colouring are reversed, that is, edge-colouring is polynomial and total-colouring is NP-complete. Bipartite unichord-free graphs provide a first example of such class.

Table 1 exhibits the four possible combinations of complexities of edge-colouring and total-colouring in subclasses of unichord-free graphs.

Table 1 Computational complexity of colouring problems restricted to subclasses of unichord-free graphs—star indicates the result established in the present paper

Class\Problem	Edge-colouring	Total-colouring
Unichord-free	NP-complete [8]	NP-complete [7]
Chordless	Polynomial [9]	Polynomial [9]
{Square,unichord}-free	NP-complete [8]	Polynomial [6]
Bipartite unichord-free	Polynomial [4]	NP-complete*

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